Part III <u>Numerical Methods</u>

<u>Discretization Methods</u>

discretization method = PDE formulation + approximating class

- <u>finite element method</u> (VF + FE space)

$$(FE)\quad \left[\begin{array}{l} \text{Find } u_h \in V_{h,g} = H^1_g(I) \cap \mathcal{S}^0_1(\Delta) \text{ s.t.} \\ \qquad a(u_h, v) = f(v) \quad \forall v \in V_{h,0} \end{array}\right.$$

where

$$V_{h,g} = \left\{ v \in \mathcal{S}^0_1(\Delta) \;\middle|\; v|_{\Gamma_D} = g \right\}, \quad \Gamma_D \in \{a, b\}.$$

<u>the corresponding algebraic equation</u>

- $u_h(x) = \sum_{i=1}^{N} u_i \varphi_i(x) \in V_h = \text{span}\{\varphi_i(x)\}_{i=1}^{N}$

- $f(v)$ is linear $\Leftrightarrow f(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 f(v_1) + \alpha_2 f(v_2)$

- $a(u,v)$ is bilinear $\Leftrightarrow$ $a(u,v)$ is linear w.r.t. u and v.

$$(FE) \Longrightarrow \quad f(v) = a(u_h, v) = \sum_{i=1}^{N} u_i \, a(\varphi_i, v) \qquad \forall v \in V_{h,0}$$

$$\underline{v=\varphi_j} \qquad \sum_{i=1}^{N} u_i \, a(\varphi_i, \varphi_j) = f(\varphi_j) \quad \text{for } j=1,\cdots,N.$$

$$\Rightarrow \qquad \underset{N\times N}{A}\,\underset{N\times 1}{U} = \underset{N\times 1}{F}$$

where $U = (u_1, \cdots, u_N)^T$, $F = (f(\varphi_1), \cdots, f(\varphi_N))^T$,

$$A = (a_{ij})_{N\times N} \quad \text{with } a_{ij} = a(\varphi_j, \varphi_i).$$

<u>Questions</u>

(1) existence, uniqueness, stability ?

(2) error estimate : $\| u - u_h \|_a \le$?

(3) algebraic solvers ?

(4) how to choose the partition Δ ?

- <u>NN method</u>

$$\mathcal{M}_n = \left\{ c_0 + \sum_{i=1}^{n} c_i \sigma(x - b_i) \,\middle|\, c_i, b_i \in R \right\}$$

<u>Properties of $\mathcal{M}_n$</u>

(1) $n+1$ linear parameters $\{c_i\}_{i=0}^{n}$

n nonlinear parameters $\{b_i\}_{i=1}^{n}$

(2) Assume that $\{b_i\}_{i=1}^{n}$ is fixed and distinct.

Then $\{1, \sigma(x-b_1), \cdots, \sigma(x-b_n)\}$ is l. indep.

<u>Proof</u> (Liu-Cai-Chen, CAMA 2022)

- $\{1, \sigma(x-b_1)\}$ is l. indep because $\sigma(x-b_1) \equiv 0 \quad \forall x \in (-\infty, b_1)$
- assume $\{1, \sigma(x-b_1), \cdots, \sigma(x-b_k)\}$ are l. indep.
- $c_0 + \sum_{i=1}^{k+1} c_i \sigma(x-b_i) \equiv 0 \implies c_0 + \sum_{i=1}^{k} c_i \sigma(x-b_i) \equiv 0 \quad \forall x < b_{k+1} \Rightarrow c_i = 0$ #

(3) $\mathcal{M}_n$ is a set but not a linear space.

Let $v_k(x) = c_0^{(k)} + \sum_i c_i^{(k)} \sigma(x - b_i^{(k)})$ for $k = 1, 2$.

assume that there exists j s.t. $b_j^{(1)} \neq b_j^{(2)}$

$\Longrightarrow$ $v_1 + v_2$ has at least $n+1$ breaking pts

$\Longrightarrow$ $v_1 + v_2 \notin \mathcal{M}_n$.

- <u>NN-based methods</u>

(1) <u>based on the energy minimization formulation</u>

deep Ritz, finite neuron, Ritz net, ...

(2) <u>based on the least-squares formulations</u>

PINN, LSNN, ...

<u>least-squares neural network (LSNN) method</u>

Find $u_n \in \mathcal{M}_n$ s.t.

$$\mathcal{L}(u_n; f) = \min_{v \in \mathcal{M}_n} \mathcal{L}(v; f).$$

Homework Problems (Brenner & Scott)

Chapter 0 #2, 3, 8

#2. Derive variational formulation of BVP $\begin{cases} -u'' + u = f & \text{in } (0,1) \\ u(0) = u(1) = 0. \end{cases}$

#3. Derive variational formulation of BVP $\begin{cases} -u'' = f & \text{in } (0,1) \\ u'(0) = 0, \ u'(1) = 0 \end{cases}$

Explain why (BVP) and (VP) are not well-posed.

#8. Prove that (BVP) $\begin{cases} -u'' = f & \text{in } (0,1) \\ u(0) = 0, \ u'(1) = 0 \end{cases}$ has a solution $u \in C^2([0,1])$ provided $f \in C^0([0,1])$.

(Hint: $u(x) = \int_0^x \left(\int_s^1 f(t)\,dt \right) ds$)

Chapter 1 #1, 3

#1. Suppose that Ω is bounded and that $1 \le p \le q \le \infty$.

Prove that $L^q(\Omega) \subset L^p(\Omega)$.

#3. Suppose that Ω is bounded and that $\lim_{j\to\infty} \| f_j - f \|_{L^p(\Omega)}$.

Prove $\lim_{j\to\infty} \int_\Omega f_j(x)\,dx = \int_\Omega f(x)\,dx$.